Rossmoyne Senior High School

Semester Two Examination, 2020

Question/Answer booklet

MATHEMATICS METHODS UNITS 1&2

Section Two: Calculator-assumed

SOLUTIONS

WA student number: In figures ☐☐ ☐☐☐ ☐☐☐

In words ____________________

Your name ____________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Number of additional answer booklets used (if applicable): ☐

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of examinations are detailed in the school handbook. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

SN085-162-4

Section Two: Calculator-assumed 65% (98 Marks)

This section has **thirteen** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 100 minutes.

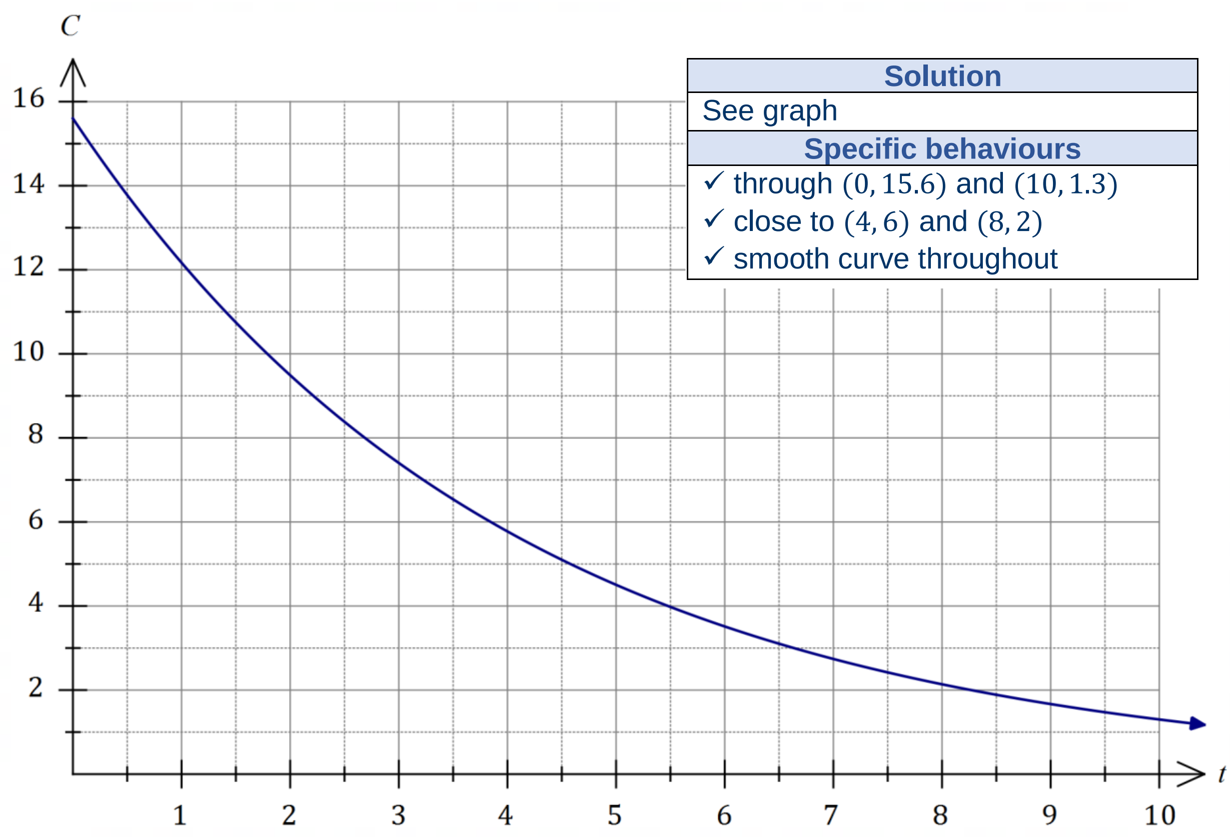
Question 9 (6 marks)

The cost, C dollars, for a gigabyte of computer memory between the end of year 2007 ($t = 0$) and the end of year 2017 ($t = 10$) can be modelled by the equation $C = 15.6(0.78)^t$.

(a) Calculate C at the end of year 2009. (1 mark)

Solution
$C(2) = \$9.49$
Specific behaviours
✓ correct cost, to nearest cent

(b) Draw the graph of C against t on the axes below. (3 marks)



Solution
See graph
Specific behaviours
✓ through $(0, 15.6)$ and $(10, 1.3)$
✓ close to $(4, 6)$ and $(8, 2)$
✓ smooth curve throughout

(c) Assuming that the model continues to be valid, during which year will the cost of computer memory first fall below 50 cents per gigabyte? (2 marks)

Solution
$C(t) = 0.5 \Rightarrow t = 13.8$ Hence during the year $2007 + 14 = 2021$
Specific behaviours
✓ correct value of t
✓ correct year

Question 10 **(5 marks)**

For the events A and B, $P(A) = 0.52$ and $P(B) = 0.25$.

Determine $P(A \cup B)$ when

(a) A and B are mutually exclusive. (1 mark)

Solution
$P(A \cup B) = 0.52 + 0.25$ $= 0.77$
Specific behaviours
✓ correct probability

(b) $P(A \cap \bar{B}) = 0.33$. (1 mark)

Solution
$P(A \cup B) = 0.33 + 0.25$ $= 0.58$
Specific behaviours
✓ correct probability

(c) $P(\bar{A} \cap \bar{B}) = 0.33$. (1 mark)

Solution
$P(A \cup B) = 1 - 0.33$ $= 0.67$
Specific behaviours
✓ correct probability

(d) A and B are independent. (2 marks)

Solution
$P(A \cap B) = 0.52 \times 0.25 = 0.13$ $P(A \cup B) = 0.52 + 0.25 - 0.13$ $= 0.64$
Specific behaviours
✓ $P(A \cap B)$ ✓ correct probability

Question 11 **(7 marks)**

A set of 175 undergraduates were asked to choose their electives for the following year. 85 chose calculus, 58 chose statistics and 67 chose neither calculus nor statistics.

(a) Determine how many of the undergraduates chose both calculus and statistics. (2 marks)

Solution
$n(C \cup S) = 175 - 67 = 108$ $n(C \cap S) = 85 + 58 - 108 = 35$ Hence 35 chose both electives.
Specific behaviours
✓ indicates union of sets ✓ correct number

(b) Determine the probability that a randomly chosen undergraduate from the set chose

(i) statistics. (1 mark)

Solution
$P(S) = \frac{58}{175} \approx 0.3314$
Specific behaviours
✓ correct probability

(ii) statistics but not calculus. (1 mark)

Solution
$P(S \cap \bar{C}) = \frac{58 - 35}{175} = \frac{23}{175} \approx 0.1314$
Specific behaviours
✓ correct probability

(iii) statistics given that they chose calculus. (1 mark)

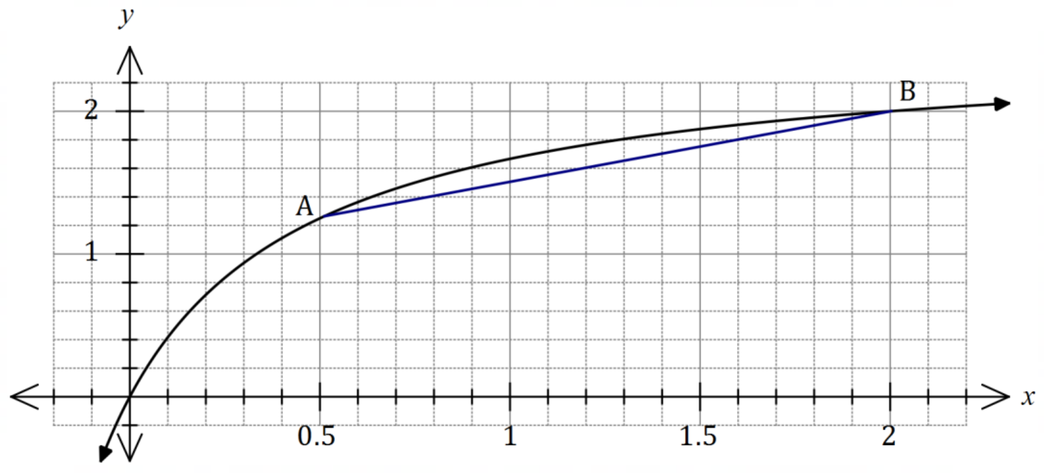
Solution
$P(S\|C) = \frac{35}{85} = \frac{7}{17} \approx 0.4118$
Specific behaviours
✓ correct probability

(c) Use your answers above to explain whether the choice of statistics and calculus electives is independent for these undergraduates. (2 marks)

Solution
Choice is not independent, as $P(S) \neq P(S\|C)$. *(Can use other independence conditions)*
Specific behaviours
✓ states not independent ✓ explanation using existing probabilities

Question 12 **(8 marks)**

Part of the graph of $y = f(x)$ is shown below, where $f(x) = \dfrac{5x}{2x+1}$.



Points A and B lie on the curve and have x-coordinates of 0.5 and 2 respectively.

(a) Draw the chord to the curve between A and B on the axes above and determine the gradient of this chord. (3 marks)

Solution
$m = \dfrac{f(2) - f(0.5)}{2 - 0.5} = \dfrac{2 - 1.25}{1.5} = 0.5$
Specific behaviours
✓ draws chord on graph ✓ correct y-values ✓ correct gradient

Point C, with an x-coordinate of $0.5 + h$, lies on the curve between A and B. The gradient of the chord AC is m_{AC}.

(b) Calculate m_{AC} for the values of h shown in the table below, recording the gradients in the table to 3 decimal places. (3 marks)

h	1	0.5	0.1	0.05	0.01
m_{AC}	0.625	$\mathbf{0.833}$	$\mathbf{1.136}$	$\mathbf{1.190}$	$\mathbf{1.238}$

Solution
See table
Specific behaviours
✓ one correct gradient ✓ at least three correct gradients ✓ all correct gradients (just comment on incorrect rounding)

(c) Determine a limiting value for m_{AC} as h becomes very close to 0 and state what feature of the graph of $y = f(x)$ this value represents. (2 marks)

Solution
As $h \to 0$ then $m_{AC} \to 1.25$. This is the gradient of $y = f(x)$ at the point A.
Specific behaviours
✓ limiting value ✓ states gradient at the point A / accept rate of change of $f(x)$ at $x = 1$

Question 13 **(7 marks)**

In flat rate depreciation, the value of an asset is depreciated by a fixed amount each year. Using the flat rate model, the value V_n of a machine in dollars after n years is given by $V_{n+1} = V_n - 350$, $V_0 = 4\,550$.

(a) Determine

(i) the value of the machine after 3 years. (1 mark)

Solution
$V_3 = \$3500$
Specific behaviours
✓ correct value

(ii) the number of years until the machine has no value. (1 mark)

Solution
$V_n = 0 \Rightarrow n = 13$ years
Specific behaviours
✓ correct number

Using flat rate depreciation, the value of another machine after 4 years will be \$2 940 and after a further 12 years it will become worthless. The value T_n of this machine after n years can be modelled using $T_n = an + b$, where a and b are constants.

(b) Determine the value of a and the value of b. (3 marks)

Solution
$2\,940 \div 12 = 245$ $T_0 = 2\,940 + 4 \times 245 = 3\,920$ $T_n = 3\,920 - 245n$ Hence $a = -245$ and $b = 3\,920$.
Specific behaviours
✓ annual loss in value ✓ initial value ✓ clearly states each value

(c) Given that both machines begin to depreciate at the same time, determine the number of years until the machines have the same value and state what this value is. (2 marks)

Solution
Using a table, the values are both \$2 450 after 6 years.
Specific behaviours
✓ years ✓ value

Question 14 **(8 marks)**

A farm grows two varieties of apples - Yates and Jonagold. 28% of all apples are grown in orchard A, 34% in orchard B and the remainder in orchard C. The proportion of Yates apples that are grown in orchards A, B and C are $55\%, 40\%$ and 25% respectively. After harvesting, the farm stores all the apples together in a large silo before using them to make apple juice.

(a) Determine the probability that an apple chosen at random from the silo is

(i) a Yates grown in orchard C. (2 marks)

Solution
$P(C) = 1 - 0.28 - 0.34 = 0.38$ $P(C \cap Y) = 0.38 \times 0.25$ $= 0.095$
Specific behaviours
✓ proportion grown in C ✓ correct probability

(ii) a Jonagold. (3 marks)

Solution
$P(A \cap J) = 0.28 \times 0.45 = 0.126$ $P(B \cap J) = 0.34 \times 0.6 = 0.204$ $P(C \cap J) = 0.38 \times 0.75 = 0.285$ $P(J) = 0.126 + 0.204 + 0.285$ $= 0.615$
Specific behaviours
✓ at least one correct proportion ✓ all correct proportions ✓ correct probability

(b) Given that an apple selected at random is a Yates, determine the probability that it was grown in orchard B. (3 marks)

Solution
$P(Y) = 1 - 0.615 = 0.385$ $P(B \cap Y) = 0.34 \times 0.4 = 0.136$ $P(B\|Y) = \dfrac{0.136}{0.385} \approx 0.353$
Specific behaviours
✓ $P(Y)$ ✓ $P(B \cap Y)$ ✓ correct probability

Question 15 **(6 marks)**

A farmer was treating a large area of land for an invasive weed. The area treated on the first day was 275 m^2. Over the following months more resources were utilised so that the area treated each day was 7.5% more than the previous day.

(a) Determine the area treated on the 28^{th} day. (2 marks)

Solution
$T_{28} = 275(1.075)^{(28-1)}$ $= 1\,938\ \text{m}^2$
Specific behaviours
✓ use general formula / recursive rule ✓ correct area

The cost of the treatment was 35.8 cents per square metre.

(b) On which day did the cost of the days treatment first exceed $10 000? (2 marks)

Solution
$C_n = 0.358 \times 275(1.075)^{(n-1)}$ $= 98.45(1.075)^{(n-1)}$ $98.45(1.075)^{(n-1)} \geq 10\,000$ $n \geq 65$ On day 65.
Specific behaviours
✓ adjusts sequence/indicates equation to solve ✓ correct day

Solution
$Cost = \$10\,000\ /\ 0.358 = \$27932.96$ Uses table recursive on CAS On day 65.
Specific behaviours
✓ correct cost ✓ correct day

(c) Determine the total cost of the first 15 days of treatment. (2 marks)

Solution
$S_{15} = \dfrac{98.45(1-1.075^{15})}{1-1.075}$ $= \$2\,571.35$
Specific behaviours
✓ indicates use of sum formula ✓ total cost

Solution
Uses table recursive on CAS $S_{15} = 7182.55$ Cost = 7182.55 x 0.358 = $2571.35
Specific behaviours
✓ correct S_{15} ✓ correct cost

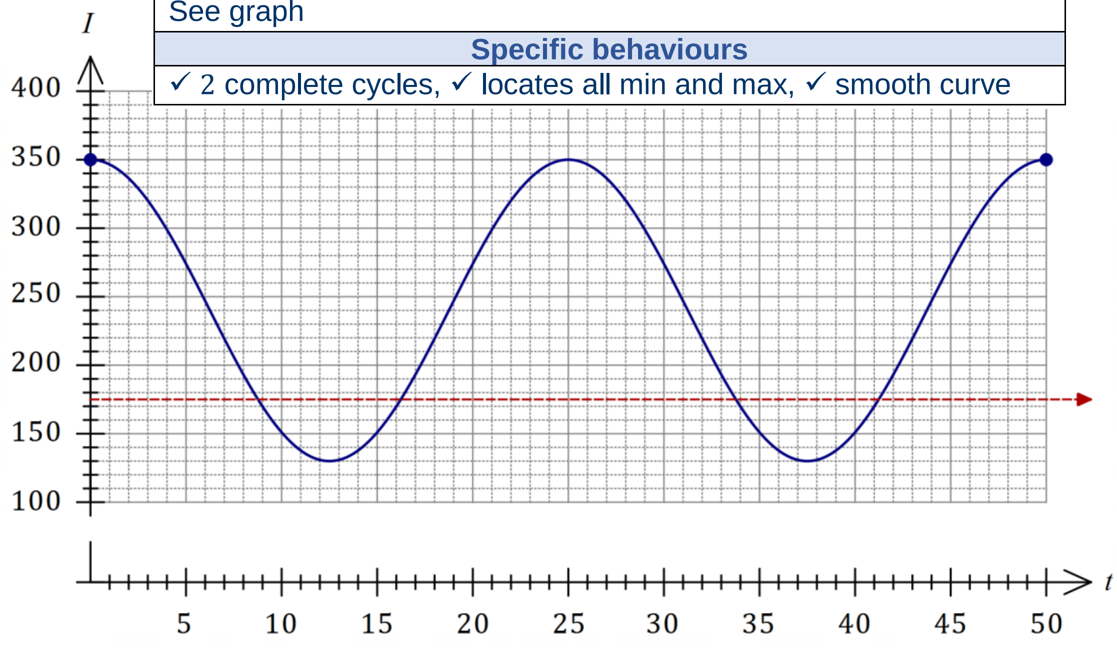
Question 16 **(6 marks)**

When an alternating current is used to power a light globe, the intensity of light emitted from the globe, I lumens, varies with time t milliseconds and can be modelled by the formula

$$I = 240 + 110\cos(0.08\pi t).$$

(a) Draw the graph of I against t on the axes below for $0 \le t \le 50$. (3 marks)

Solution
See graph
Specific behaviours
✓ 2 complete cycles, ✓ locates all min and max, ✓ smooth curve



(b) State the period of I. (1 mark)

Solution
Period is 25 milliseconds
Specific behaviours
✓ correct period

(c) Determine the percentage (to the nearest percent) of each cycle that the intensity of light is below 175 lumen.

(2 marks)

Solution
$I = 175 \Rightarrow t = 16.235, 8.765$ $16.235 - 8.765 = 7.47$ $\dfrac{7.47}{25} \times 100 \approx 30\%$
Specific behaviours
✓ indicates interval   ✓ correct percentage, just comment on rounding

Question 17 **(9 marks)**

(a) Point $A(11, -5)$ lies on the circumference of a circle with centre $(-4, 3)$. Determine the equation of the circle. (3 marks)

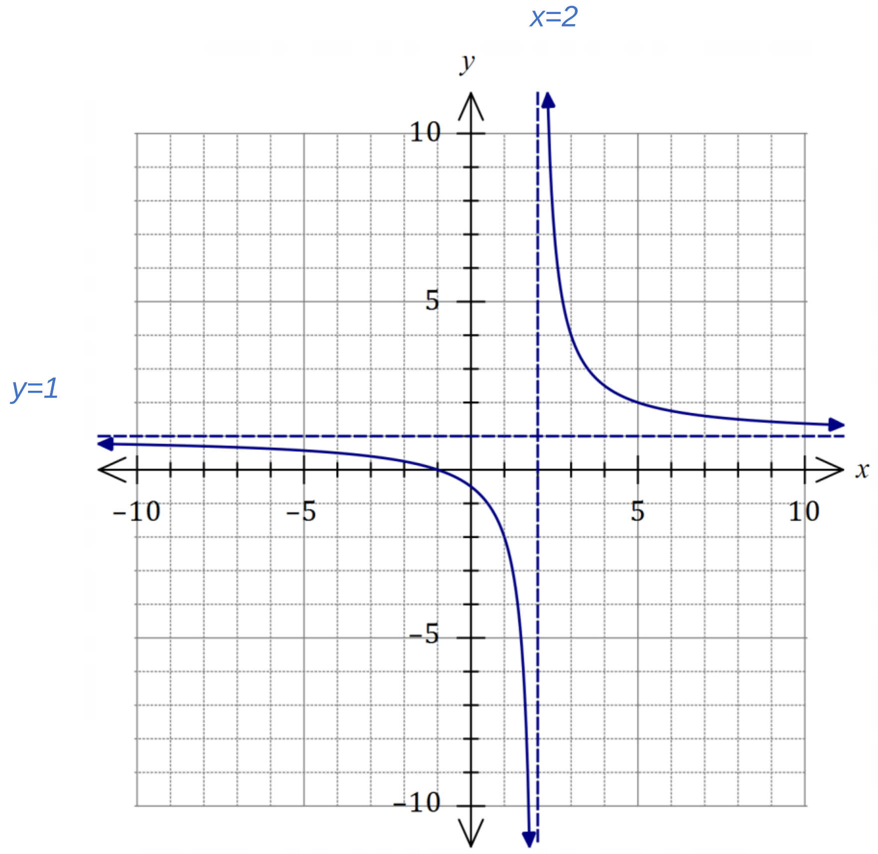
Solution
$(x+4)^2 + (y-3)^2 = k$ Using given point: $15^2 + (-8)^2 = k \Rightarrow k = 289 \ (= 17^2)$ Equation: $(x+4)^2 + (y-3)^2 = 289$
Specific behaviours
✓ forms equation using centre ✓ uses points to find k value or correct radius ✓ correct equation (any form)

(b) The graph of $y = 1 + \frac{a}{x+b}$ passes through the points $(1, -2)$ and $(3, 4)$.

(i) Determine the value of each of the integer constants a and b. (3 marks)

Solution
$-2 = 1 + \frac{a}{1+b}, \quad 4 = 1 + \frac{a}{3+b}$ Solve simultaneously for $a = 3, b = -2$.
Specific behaviours
✓ uses points to form two equations ✓ correct a ✓ correct b

(ii) Draw the graph of $y = 1 + \dfrac{a}{x+b}$ on the axes below, clearly labelling any asymptotes. (3 marks)



Solution
See graph
Specific behaviours
✓ both asymptotes drawn and labelled ✓ LHS, smooth curve, through $(1, -2)$ ✓ RHS, smooth curve, through $(3, 4)$

Question 18 **(13 marks)**

A small body is moving in a straight line. Relative to a fixed point O, it has a displacement of x cm at time t seconds given by

$$x(t) = 2t^3 - 19t^2 + 52t - 35, \qquad 0 \le t \le 6.$$

(a) Obtain an expression for the velocity of the body in the form $v(t) = (at + b)(ct + d)$, where a, b, c and d are integer constants. (3 marks)

Solution
$v(t) = \frac{dx}{dt}$ $= 6t^2 - 38t + 52$ $= 2(t-2)(3t-13)$ $= (2t-4)(3t-13)$ (or $= (t-2)(6t-26)$)
Specific behaviours
✓ uses derivative of $x(t)$ to find $v(t)$ ✓ correct derivative   ✓ factors into required form

(b) Determine

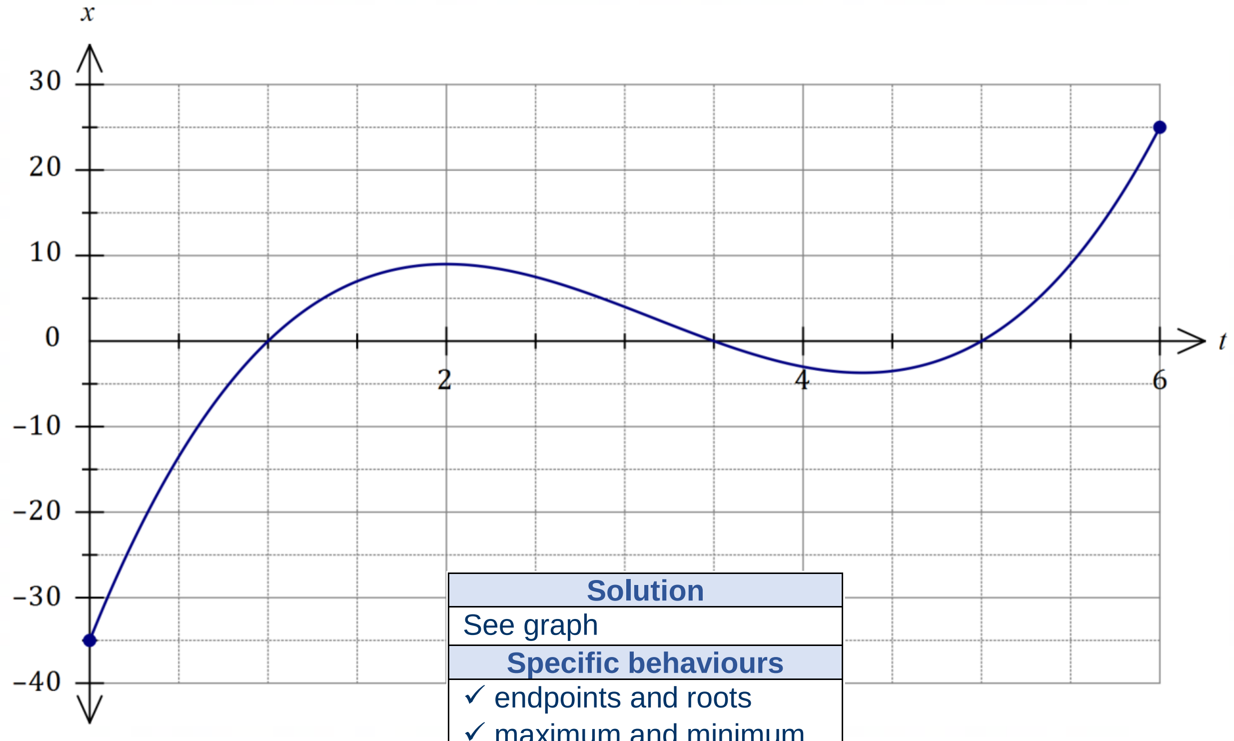
(i) the initial velocity of the body. (1 mark)

Solution
$v(0) = 52 \text{ cm/s}$
Specific behaviours
✓ correct velocity

(ii) the displacement of the body at the instant(s) that it is stationary. (3 marks)

Solution
$v(t) = 0 \Rightarrow t = 2, t = \frac{13}{3}$ $x(2) = 9 \text{ cm}$ $x\left(\frac{13}{3}\right) = -\frac{100}{27} \approx -3.70 \text{ cm}$
Specific behaviours
✓ times when stationary   ✓ one correct displacement   ✓ both correct, **with units**

(c) Use the axes below to sketch the displacement of the body over the given domain. (3 marks)



Solution
See graph
Specific behaviours
✓ endpoints and roots ✓ maximum and minimum ✓ smooth curve

(d) State the number of times the body passed through O and determine the minimum speed and maximum speed of the body as it passed through this point. (3 marks)

Solution
Passes through O when $t = 1, 3.5, 5$ s - on 3 occasions. $v(1) = 20$ $v(3.5) = -7.5$ $v(5) = 12$ Hence minimum speed is 7.5 cm/s and maximum speed is 20 cm/s. Speed must be positive
Specific behaviours
✓ correct number of times ✓ minimum speed ✓ maximum speed

Question 19 **(6 marks)**

A reader bought 15 different novels, planning to read a selection of them when on holiday.

(a) Determine the number of different combinations of novels the reader could choose from if they select

(i) five novels. (1 mark)

Solution
$\binom{15}{5} = 3\ 003$ combinations
Specific behaviours
✓ correct number

(ii) four or five novels. (2 marks)

Solution
$\binom{15}{4} + \binom{15}{5} = 1365 + 3003$ $= 4\ 368$ combinations
Specific behaviours
✓ ways to choose four   ✓ correct number

Six of the 15 different novels are by the author Rowe.

(b) Determine the number of different combinations the reader could choose from if they select 5 novels of which

(i) none of the novels selected are by Rowe. (1 mark)

Solution
Must choose from 9 not by Rowe: $\binom{9}{5} = 126$
Specific behaviours
✓ correct number

(ii) three of the novels selected are by Rowe. (2 marks)

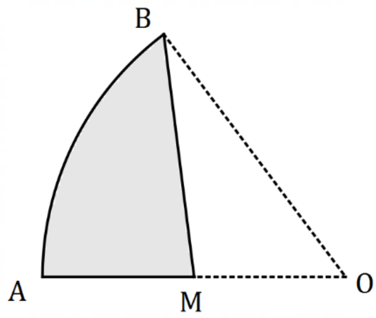
Solution
$\binom{9}{2}\binom{6}{3} = 36 \times 20 = 720$
Specific behaviours
✓ one selection correct   ✓ correct number

Question 20 **(9 marks)**

The shaded region ABM in the diagram is a canvas awning and is part of a sector of circle OAB with centre O and radius 4.2 m.

BM is a straight line from B to M, the midpoint of OA.

The size of $\angle AOB$ is 0.9 radians.



(a) Determine the area of sector OAB. (2 mark)

Solution
$A_{OAB} = 0.5(4.2)^2 \times 0.9 = 7.938\ \text{m}^2$
Specific behaviours
✓ shows use of area formula ✓ correct sector area

(b) Determine the area of the canvas awning. (3 marks)

Solution
$A_{OMB} = 0.5(4.2)(2.1)\sin(0.9)$ $= 3.454$ $A_{ABM} = A_{OAB} - A_{OMB}$ $= 7.938 - 3.454$ $= 4.484\ \text{m}^2$
Specific behaviours
✓ indicates use of difference of areas ✓ area of triangle ✓ correct area

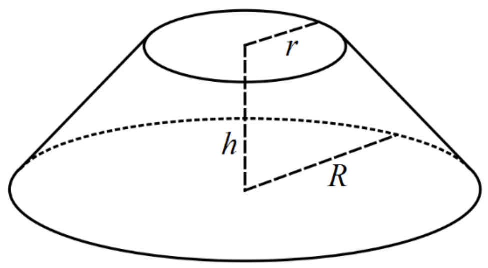
(c) The edge of the canvas is to be reinforced with thin wire. Determine the length of wire required. (4 marks)

Solution
$BM^2 = 4.2^2 + 2.1^2 - 2(4.2)(2.1)\cos(0.9)$ $BM = 3.329$ $\text{Arc}_{AB} = 3.2 \times 1.3 = 3.78$ $L = 2.1 + 3.329 + 3.78$ $= 9.21\ \text{m}$
Specific behaviours
✓ uses cosine rule for BM ✓ length of BM ✓ arc length AB ✓ correct length

(or Arc_{AB}=4.2 x 0.9 =3.78)

Question 21 **(8 marks)**

The frustum shown at right is a truncated right cone.



The volume of such a solid is $V = \frac{\pi h}{3}(r^2 + R^2 + rR)$, where r is the radius of the smaller circle, R is the radius of the larger circle and h is the perpendicular distance between the two parallel circles.

Consider frustum F where $r = x$ cm, $R = 3r$ and $r + h = 36$ cm.

(a) Show that the volume of frustum F is $156\pi x^2 - \frac{13\pi x^3}{3}$ cm^3. (3 marks)

Solution
$r = x, \quad R = 3x, \quad h = 36 - x$ $\therefore V = \frac{\pi(36 - x)}{3}\left(x^2 + (3x)^2 + x(3x)\right)$ $= \left(12\pi - \frac{\pi x}{3}\right)(13x^2)$ $= 156\pi x^2 - \frac{13\pi x^3}{3}$
Specific behaviours
✓ expresses r, R and h in terms of x (must)   ✓ substitutes and simplifies $(r^2 + R^2 + rR)$ term   ✓ clear steps to obtain final expression

(b) Use a calculus method to determine the value of x that maximises the volume of the frustum F, state this maximum volume and verify that it is indeed a maximum. (5 marks)

Solution
$\frac{dV}{dx} = 312\pi x - 13\pi x^2$ Derivative is zero when: $\pi x(312 - 13x) = 0$ $x = 0, x = 24$ $V(24) = 29952\pi \approx 94\,097$ Check Max $at\ x = 24, \quad \frac{d^2V}{dx^2} < 0 \ \therefore\ max$ Maximum volume of frustum is 94 097 cm^3 when $x = 24$ cm.
Specific behaviours
✓ derivative and equates to zero   ✓ solves to 2 solutions   ✓ states root of derivative for maximum volume   ✓ states maximum volume ✓ checks max using sign test or ...

End of questions

Supplementary page

Question number: _________